Iterated Differential Forms I: Tensors

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February 2, 2008

Abstract

This note is the first in a series of short communications dedicated to general theory and some applications of iterated differential forms. Both are developed either in the "classical"context, or in the "quantistic"one, i.e., of Secondary Calculus (see [1], [2]). Detailed expositions containing proofs of the announced results will be appearing in due course.

With iterated forms we solve the problem of a *conceptual foundation of tensor calculus*. In particular, we show that covariant tensors are differential forms over a certain graded commutative algebra called the algebra of iterated differential forms. From one side, this interpretation extends noteworthy frames of the traditional tensor calculus and enriches it by numerous new natural operators. On the other side, it allows various generalizations of tensor calculus, the most important of which is that to secondary ("quantized") calculus. In particular, this leads to an unified solution of the secondarization ("quantization") problem (see [1], [2]) for arbitrary tensors.

In this communication the algebra of iterated (differential) forms over an arbitrary (graded) commutative algebra is defined. It is also shown how tensors on a (smooth) manifold M are naturally interpreted as iterated differential forms over the algebra $C^\infty(M)$.

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According to the original definition a tensor on a manifold M is a system of suitably indexed functions, called components, associated with a local chart, which change according to a certain rule when passing to another chart. The modern definition presents a tensor to be a $C^\infty(M)$–multilinear and $C^\infty(M)$–valued function in a number of variables that are either vector fields, or differential 1–forms on M. These definitions are manifestly descriptive. For instance, on the basis of any of them it is not possible to understand why the natural exterior differential d is defined on skew–symmetric covariant tensors, i.e., differential forms, but not on symmetric ones. Or, why a natural connection, namely, that of Levi–Civita, is associated with (non–degenerate) symmetric covariant 2–tensors, but not with skew–symmetric ones, etc. According to [3] the meaning of "controvariant"objects of differential calculus, say, vector fields, or general differential operators, is given by *functors of differential calculus*. For instance, the derivation functor D corresponds to vector fields and Diff_k to k–th order differential operators. On the other hand, covariant objects are related to representing such functors objects. For instance, differential i–forms and k–jets are elements of modules that represent functors D_i and Diff_k, respectively. Concerning conceptual definition of a species of covariant tensors the problem is to attribute them to the module representing a certain functor of differential calculus. What makes this problem not very banal is that such a *direct* attribution is not, generally, possible. This is, for instance, the case of symmetric tensors. On the contrary, skew-symmetric tensors, i.e., differential forms, allow such one and the corresponding functors are D_i's. We overcome this difficulty by looking for the necessary *direct attribution* not over the original ground commutative algebra, but over another one, naturally associated with the former.

Below this idea is realized by passing to the filtered graded commutative algebra $A = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_k \subset \cdots \subset \Lambda_\infty$, where A is the ground algebra and Λ_k is the algebra of differential forms over the algebra Λ_{k-1}. This way we respond the question: what properly are covariant tensors?

In the subsequent note the first application of the theory sketched here to Riemannian geometry will be given. In particular, the nature of the Levi–Civita connection will be clarified and answered the question posed above.

1 Differential calculus over graded algebras

A pair $\mathcal{G} = (G, \mu)$, G being an abelian group and $\mu : G \times G \longrightarrow \mathbb{Z}_2$ a symmetric $\mathbb{Z}$–bilinear map, is a *grading group*. Below $\Bbbk$ stands for a field of zero–characteristic and we use $g{\cdot}h$ for $\mu(g,h)$. The category of unitary, $\mathcal{G}$–graded, associative, graded–commutative $\Bbbk$–algebras is denoted by $\mathbf{Alg}_\Bbbk^\mathcal{G}$. If $A \in \mathbf{Alg}_\Bbbk^\mathcal{G}$ and $a \in A$ is homogeneous, then $|a| \in G$ denotes the degree of a. Recall that graded–commutativity means that $ab = (-1)^{|a|\cdot|b|}ba$ for all homogeneous elements $a, b \in A$. In the sequel we adopt the following convention:

in the exponent of (-1) the symbol, say, σ, denoting a homogeneous element is used as a substitute of $|\sigma|$. For instance, $(-1)^{a\cdot b}$ means $(-1)^{|a|\cdot|b|}$. The differential calculus over an algebra $A \in \mathbf{Alg}_{\Bbbk}^{\mathcal{G}}$ is introduced along the lines of [3, 4, 5, 6] where the reader will find further details.

Denote by $\mathbf{Mod}_A^{\mathcal{G}}$ the category of $\mathcal{G}$–graded A–modules and by $\mathrm{D}_A : \mathbf{Mod}_A^{\mathcal{G}} \longrightarrow \mathbf{Mod}_A^{\mathcal{G}}$ the functor associating with $P \in \mathbf{Mod}_A^{\mathcal{G}}$ the graded A–module $\mathrm{D}_A(P)$ of P–*valued derivations* of A. $\mathrm{D}_A(A)$ is a graded $\Bbbk$–Lie algebra with respect to the graded commutator. Functors $\mathrm{D}_k : \mathbf{Mod}_A^{\mathcal{G}} \longrightarrow \mathbf{Mod}_A^{\mathcal{G}}$, $k \in \mathbb{N}$ are defined in [3, 4, 5]. Recall that an element $\nabla \in \mathrm{D}_k(P)$ may be viewed as a graded skew–symmetric P–valued multi–derivation of A of multiplicity k. In particular, $\mathrm{D}_1 = \mathrm{D}_A$. Denote by $\Lambda^k(A)$ the graded A–module of differential k–forms over A and by $\Lambda^k(A)^g$, $g \in G$, its homogeneous component of grade g.

The direct sum $\Lambda(A) = \bigoplus_{k=0}^{\infty} \Lambda^k(A)$ has a natural structure of an unitary, $\mathcal{G} \oplus \mathbb{Z}$–graded, associative, graded–commutative $\Bbbk$–algebra which is called the algebra of *differential forms over* A. It is naturally isomorphic to the $\mathcal{G}$–exterior algebra $\bigwedge^* \Lambda^1(A)$. Thus, any $\sigma \in \Lambda(A)$ is of the form

$$\sigma = \sum a_{\alpha_1 \cdots \alpha_k} db_{\alpha_1} \wedge \cdots \wedge db_{\alpha_k}, \tag{1}$$

for some $a_{\alpha_1 \ldots \alpha_k}, b_{\alpha_1}, \ldots b_{\alpha_k} \in A$.

For any $\sigma \in \Lambda^k(A)^g$ put $|\sigma| = (g,k) \in G \oplus \mathbb{Z}$. Then, $\sigma \wedge \rho = (-1)^{\sigma \cdot \rho} \rho \wedge \sigma$ for any homogeneous elements $\sigma, \rho \in \Lambda(A)$, where $(g,k)\cdot(h,l) \stackrel{\text{def}}{=} g \cdot h + [kl]_2 \in \mathbb{Z}_2$.

The *exterior differential* in $\Lambda(A)$ will be denoted by $d : \Lambda(A) \longrightarrow \Lambda(A)$. It is a graded $\Lambda(A)$–derivation of bi–degree $|d| = (0,1) \in G \oplus \mathbb{Z}$ and $d^2 = 0$. So, $(\Lambda(A), d)$ is a $\mathcal{G} \oplus \mathbb{Z}$–graded differential algebra.

The correspondence $A \longmapsto (\Lambda(A), d)$ is a functor from the category $\mathbf{Alg}_{\Bbbk}^{\mathcal{G}}$ to the category $\mathbf{dAlg}_{\Bbbk}^{\mathcal{G}\oplus\mathbb{Z}}$ of unitary, $\mathcal{G} \oplus \mathbb{Z}$–graded, associative, graded–commutative, differential $\Bbbk$–algebras. In particular, if $A, A' \in \mathbf{Alg}_{\Bbbk}^{\mathcal{G}}$, then a morphism $\phi : A \longrightarrow A'$ is extended to a morphism $\Lambda(\phi) : \Lambda(A) \longrightarrow \Lambda(A')$ compatible with the exterior differential, i.e., $d \circ \Lambda(\phi) = \Lambda(\phi) \circ d$. If $\sigma \in \Lambda(A)$ is of the form (1), then $\Lambda(\phi)(\sigma) = \sum \phi(a_{\alpha_1 \cdots \alpha_k}) d(\phi(b_{\alpha_1})) \wedge \cdots \wedge d(\phi(b_{\alpha_k}))$.

The insertion of a derivation $X \in \mathrm{D}_A(A)$ operator will be denoted by $i_X : \Lambda(A) \longrightarrow \Lambda(A)$. It is a $\Lambda(A)$–derivation of bi–degree $(|X|, -1)$. The Lie derivative along $X \in \mathrm{D}_A(A)$ is defined by $\mathcal{L}_X = [i_X, d]$. It is a graded $\Lambda(A)$–derivation of bi–degree $(|X|, 0)$ and extends X to $\Lambda(A)$.

For any $X, Y \in \mathrm{D}_A(A)$ the following (graded) commutation relations hold:

$$[i_X, i_Y] = [\mathcal{L}_X, d] = 0, \quad [i_X, \mathcal{L}_Y] = i_{[X,Y]}, \quad [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}. \tag{2}$$

2 Iterated differential forms

In what follows we put $\Lambda \equiv \Lambda(A)$. So, Λ is an unitary, graded, associative, graded–commutative algebra and all constructions of the previous section are applied to Λ and so on. This leads to the following [7]

Definition 1 *Given* $k \in \mathbb{N}$ *the* algebra Λ_k *of* k–times iterated differential forms over A *is defined inductively as* $\Lambda_k = \Lambda(\Lambda_{k-1})$, *by starting with* $\Lambda_0 = A$. *The exterior differential* d_k *in* Λ_k $(d_1 = d)$ *is called* k–th iterated exterior differential.

So, (Λ_k, d_k) is a differential, $\mathcal{G} \oplus \mathbb{Z}^k$–graded commutative algebra. Natural inclusions $\Lambda_{k-1} \subset \Lambda(\Lambda_{k-1}) = \Lambda_k$ define the filtered algebra $\Lambda_\infty : \Lambda_0 \equiv A \subset \Lambda_1 \equiv \Lambda \subset \Lambda_2 \subset \cdots \subset \Lambda_k \subset \cdots \subset \Lambda_\infty$, where $\Lambda_\infty \equiv \bigcup_k \Lambda_k$. Λ_∞ is a $\mathcal{G} \oplus \mathbb{Z}^\infty$–graded commutative algebra. If $(g, K) \in G \oplus \mathbb{Z}^k$, denote by $\Lambda_k^{(g,K)} \subset \Lambda_k$ the homogeneous component of grade (g, K). Put also $\Lambda_k^K \stackrel{\text{def}}{=} \bigoplus_{g \in G} \Lambda_k^{(g,K)} \subset \Lambda_k$.

Definition 2 Λ_∞ *is called the* algebra of iterated forms over A.

The operator of insertion of $\nabla \in \mathrm{D}_{\Lambda_k}(\Lambda_k)$ into forms $\Lambda(\Lambda_k) = \Lambda_{k+1}$ over Λ_k is denoted by $i_\nabla^{(k+1)} \in \mathrm{D}_{\Lambda_{k+1}}(\Lambda_{k+1})$. The Lie derivative $\mathcal{L}_\nabla^{(k+1)} = [i_\nabla^{(k+1)}, d_{k+1}] \in \mathrm{D}_{\Lambda_{k+1}}(\Lambda_{k+1})$ extends ∇ to Λ_{k+1}. In its turn, $\mathcal{L}_\nabla^{(k+1)}$ can be extended to Λ_{k+2} and so on up to Λ_∞. Hence any $\nabla \in \mathrm{D}_{\Lambda_k}(\Lambda_k)$ extends to a derivation of Λ_∞. This extension will be denoted by the same symbol ∇. Note that this notation is consistent with the third relation in (2).

In particular, the k–th differential d_k can be extended to a derivation of Λ_∞. Moreover, It follows from commutation relations (2) that $d_j^2 = 0$ and $[d_i, d_j] = 0$ for all i, j. This way $(\Lambda_\infty, d_1, \ldots, d_k, \ldots)$ becomes a multiple complex. Note also that for a given k the correspondence $A \longmapsto (\Lambda_\infty, d_k)$ is a functor from the category $\mathbf{Alg}_\Bbbk^{\mathcal{G}}$ to the category $\mathbf{dAlg}_\Bbbk^{\mathcal{G} \oplus \mathbb{Z}^\infty}$.

Note that $\Lambda_{k+l} = \Lambda_l(\Lambda_k)$ for any $k, l \in \mathbb{N}$. Therefore, $\Lambda_l(\Lambda_\infty) = \Lambda_\infty$, $l \in \mathbb{N}$. Indeed,

$$\Lambda_l(\Lambda_\infty) = \Lambda_l(\textstyle\bigcup_k \Lambda_k) = \bigcup_k \Lambda_l(\Lambda_k) = \bigcup_k \Lambda_{k+l} = \Lambda_\infty.$$

We express this fact by saying accordingly that Λ_∞ *is* Λ–*closed.*

Proposition 3 *There is an isomorphism* $\kappa_{(12)}$ *of the double complexes* (Λ_2, d_1, d_2) *and* (Λ_2, d_2, d_1).

Proof. The proposition is proved via the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "d_1"] \arrow[d, equal] & \Lambda_1^1 \arrow[r, hook] \arrow[d, "i_{d_2}"] & \Lambda_1 \arrow[r, "d_2"] \arrow[d, "\kappa'"] \arrow[dd, bend right, "\bigwedge^*(i_{d_2})"'] & \Lambda_2^1 \arrow[r, hook] \arrow[d, "\kappa''"] \arrow[dd, bend right, "\Lambda^1(\kappa')"'] & \Lambda_2 \arrow[d, shift left, "\kappa_{(12)}"] \arrow[dd, bend right, "\bigwedge^* \kappa''"'] \\
A \arrow[r, "d_2"] & \Lambda_2^{(0,1)} \arrow[r, hook] \arrow[dr, hook] & \Lambda_2^{(0,*)} \arrow[r, "d_1"] \arrow[dr, "d''"'] & \Lambda_2^{(1,*)} \arrow[r, hook] \arrow[dr, hook] & \Lambda_2 \arrow[u, shift left] \\
 & & \bigwedge_A^*(\Lambda_2^{(0,1)}) \arrow[u, "\wedge'"'] & \Lambda^1(\Lambda_2^{(0,*)}) \arrow[u, "i_{d''}"'] & \bigwedge^*_{\Lambda_2^{(0,*)}}(\Lambda_2^{(1,*)}) \arrow[u, "\wedge'''"']
\end{tikzcd}$$

where the following notations have been adopted. $\Lambda_2^{(m,*)} = \bigoplus_l \Lambda_2^{(m,l)}$, $m = 0, 1$, $\wedge'$ is the product in Λ_2 of elements belonging to $\Lambda_2^{(0,1)} \subset \Lambda_2$ and $\kappa' = \wedge' \circ \bigwedge^*(i_{d_2})$. Note that $\Lambda_2^{(0,*)}$ is a $\mathcal{G} \oplus \mathbb{Z}$–graded, unitary, graded–commutative algebra and $d'' : \Lambda_2^{(0,*)} \longrightarrow \Lambda^1(\Lambda_2^{(0,*)})$ is its first de Rham differential. Moreover $\kappa'' = i_{d''} \circ \Lambda^1(\kappa')$. Finally $\wedge'''$ is the product in Λ_2 of elements belonging to $\Lambda_2^{(1,*)} \subset \Lambda_2$ and $\kappa_{(12)} = \wedge''' \circ \bigwedge^* \kappa''$. It is straightforward to see that

$$\begin{aligned}
&\kappa_{(12)}(d_1 g_1 \wedge \cdots \wedge d_1 g_p \wedge d_2 h_1 \wedge \cdots \wedge d_2 h_q \wedge d_1 d_2 \ell_1 \wedge \cdots \wedge d_1 d_2 \ell_r) \\
&= d_2 g_1 \wedge \cdots \wedge d_2 g_p \wedge d_1 h_1 \wedge \cdots \wedge d_1 h_q \wedge d_1 d_2 \ell_1 \wedge \cdots \wedge d_1 d_2 \ell_r
\end{aligned}$$

for any $g_1, \ldots, g_p, h_1, \ldots, h_q, \ell_1, \ldots, \ell_r \in A$. Therefore, $\kappa_{(12)}$ is an involution of Λ_2. ∎

Let S_k be the group of permutations of $\{1, \ldots, k\}$.

Corollary 4 *For any $k \in \mathbb{N}$ and $\sigma \in S_k$ there is an isomorphism κ_σ of the multiple complexes $(\Lambda_k, d_1, \ldots, d_k)$ and $(\Lambda_k, d_{\sigma(1)}, \ldots, d_{\sigma(k)})$.*

Proof. The proof is by induction on k. Obviously, it is sufficient to prove the assertion for a transposition σ. The base of induction, $k = 2$, is provided by the above proposition. Now, suppose the corollary be true for Λ_{k-1}. In particular, for any $i, j < k$ there is an isomorphism $\kappa_{i,j}$ of the complexes (Λ_{k-1}, d_i) and (Λ_{k-1}, d_j) which commutes with all the other differentials. By extending the d_i's, $i < k$, as derivations to Λ_k one finds that the complexes (Λ_k, d_i) and (Λ_k, d_j), $i, j < k$ are isomorphic as well. Moreover, there exists an isomorphism of the complexes (Λ_k, d_{k-1}) and (Λ_k, d_k). Therefore (Λ_k, d_i) and (Λ_k, d_j) are isomorphic for any $i, j \leq k$. By abusing the notation we again denote by $\kappa_{i,j}$ such isomorphism. Since for any algebra automorphism which commutes with a given derivation X, the corresponding automorphism of the differential form algebra commutes with the Lie derivative along X, it is clear that $\kappa_{i,j}$ commutes with the differentials d_l, $l \neq i, j$. ∎

Corollary 5 *For any permutation $\sigma \in S_{\mathbb{N}}$ there exists an isomorphism κ_σ of the multiple complexes $(\Lambda_\infty, d_1, \ldots, d_k, \ldots)$ and $(\Lambda_\infty, d_{\sigma(1)}, \ldots, d_{\sigma(k)}, \ldots)$.*

Proof. Let $\sigma \in S_{\mathbb{N}}$ and $\Omega \in \Lambda_k \subset \Lambda_\infty$. Let $l = \max\{\sigma(1), \ldots, \sigma(k)\}$ and $\tilde{\sigma} \in S_l$ be such that $\tilde{\sigma}(i) = \sigma(i)$ for any $i \leq k$. Put $\kappa_\sigma(\Omega) \equiv \kappa_{\tilde{\sigma}}(\Omega)$. κ_σ does not depend on the choice of $\tilde{\sigma}$ and is therefore well-defined. Moreover, κ_σ has inverse $\kappa_{\sigma^{-1}}$. Thus it is an isomorphism of the multiple complexes $(\Lambda_\infty, d_1, \ldots, d_k, \ldots)$ and $(\Lambda_\infty, d_{\sigma(1)}, \ldots, d_{\sigma(k)}, \ldots)$. ■

The correspondence $\sigma \longmapsto \kappa_\sigma$ defines an action of the group $S_{\mathbb{N}}$ on Λ_∞.

An important fact is that the cohomology of complexes (Λ_∞, d_k), $k \in \mathbb{N}$ is "constant".

Theorem 6 *Complexes (Λ_2, d_2) and (Λ, d) are naturally homotopy equivalent.*

Proof. In the proof of proposition 3 a natural isomorphism between the complexes $(\Lambda_2^{(0,*)}, d_2)$ and (Λ, d) was established. Let $\iota : (\Lambda_2^{(0,*)}, d_2) \longrightarrow (\Lambda_2, d_2)$ be a natural embedding and $\pi : (\Lambda_2, d_2) \longrightarrow (\Lambda_2^{(0,*)}, d_2)$ a natural projection. We shall prove that the pair (ι, π) is a homotopy equivalence. Since $\pi \circ \iota = \mathrm{id}_{\Lambda_2^{(0,*)}}$ it suffices to prove that $\iota \circ \pi$ is homotopy equivalent to id_{Λ_2}. If $\Omega \in \Lambda_2^{(i,j)}$, then

$$(\mathrm{id}_{\Lambda_2} - \iota \circ \pi)(\Omega) = \begin{cases} \Omega & \text{if } i \neq 0 \\ 0 & \text{if } i = 0 \end{cases} .$$

Denote by d^0 the 0–degree component of the ordinary de Rham differential. It is a Λ^1–valued derivation of A. The corresponding insertion operator $C = i_{d^0} : \Lambda \longrightarrow \Lambda$ is a derivation of Λ and we have $C(\sigma) = s\sigma$ for $\sigma \in \Lambda^s$.

Consider the 0–degree map $i_C^{(2)} : \Lambda_2 \longrightarrow \Lambda_2$ and define the map $H_2 : \Lambda_2 \longrightarrow \Lambda_2$ by:

$$H_2(\Omega) = \begin{cases} \frac{1}{s} i_C^{(2)}(\Omega) & \text{if } s \neq 0 \\ 0 & \text{if } s = 0 \end{cases} ,$$

for $\Omega \in \Lambda_2^{(s,t)}$. Prove that H_2 is a homotopy connecting the chain maps ι and π. Indeed, if $s = 0$, then $[H_2, d_2](\Omega) = 0 = (\mathrm{id}_{\Lambda_2} - \iota \circ \pi)(\Omega)$. If $s \neq 0$, then $[H_2, d_2](\Omega) = \frac{1}{s}[i_C^{(2)}, d_2](\Omega) = \frac{1}{s}\mathcal{L}_C^{(2)}\Omega$. Show that $\mathcal{L}_C^{(2)}\Omega = s\Omega$. Assume that $\Omega = \sum \sigma_{\alpha_1 \ldots \alpha_l} \wedge d_2\sigma_{\alpha_1} \wedge \cdots \wedge d_2\sigma_{\alpha_l}$ for $\sigma_{\alpha_1 \cdots \alpha_l}, \sigma_{\alpha_1}, \ldots, \sigma_{\alpha_l} \in \Lambda$. Then

$$\begin{aligned} \mathcal{L}_C\Omega = &\sum C(\sigma_{\alpha_1 \cdots \alpha_l}) \wedge d_2\sigma_{\alpha_1} \wedge \cdots \wedge d_2\sigma_{\alpha_l} + \sum \sigma_{\alpha_1 \cdots \alpha_l} \wedge d_2C(\sigma_{\alpha_1}) \wedge \sigma_{\alpha_2} \wedge \cdots \wedge d_2\sigma_{\alpha_l} \\ &+ \cdots + \sum \sigma_{\alpha_1 \cdots \alpha_l} \wedge d_2\sigma_{\alpha_1} \wedge \cdots \wedge d_2\sigma_{\alpha_{l-1}} \wedge d_2C(\sigma_{\alpha_l}) = s\Omega. \end{aligned}$$

Therefore $[H_2, d_2](\Omega) = \Omega = (\mathrm{id}_{\Lambda_2} - \iota \circ \pi)(\Omega)$. ■

The following consequence is obvious.

Corollary 7 *for any k $H(\Lambda_\infty, d_k) \simeq H(\Lambda, d)$.*

3 Iterated differential forms over a smooth manifold

In this and the next two sections our considerations are restricted to the case $A = C^\infty(M)$, where M is an n–dimensional smooth manifold. In this situation it is convenient to pass to the category $\mathbf{Mod}_A^g$ of geometric A–modules (see, e.g., [8]) and to work with representing objects of functors of differential calculus only in this category. For instance, geometric differential forms over the algebra A are nothing but standard differential forms over the manifold M.

Denote by $\Lambda_\infty(M)$ (or simply Λ_∞) iterated geometric differential forms over $C^\infty(M)$. Their coordinate description is as follows. Let $K = \{k_1, \ldots, k_r\} \subset \mathbb{N}$ be a finite subset. For $f \in A$ put $d_K f \stackrel{\text{def}}{=} d_{k_1} \cdots d_{k_r} f$. If $x^1, \ldots, x^n$ are local coordinates on M, then $\Lambda_\infty(M)$ is locally generated as an algebra by the elements $d_K x^\mu$, $\mu = 1, \ldots, n$, $K \subset \mathbb{N}$. In particular, it is easy to see that locally

$$d_K f = \sum_{\{J_1,\ldots,J_l\}} \frac{\partial^l f}{\partial x^{\mu_1} \cdots \partial x^{\mu_l}} d_{J_1} x^{\mu_1} \wedge \cdots \wedge d_{J_l} x^{\mu_l},$$

where the sum runs over all repeated indexes and all partitions $\{J_1, \ldots, J_l\}$ of K into l parts, $1 \leq l \leq r$.

Let N be an m–dimensional manifold, $(y^1, \ldots, y^m)$ local coordinates on N and $\phi : M \longrightarrow N$, $x^\mu \longmapsto y^\alpha = \phi^\alpha(x)$, $\mu = 1, \ldots, n$, $\alpha = 1, \ldots, m$, be a smooth map. Denote by Λ'_∞ the algebra of iterated differential forms over N and consider the homomorphism $\Lambda_\infty(\phi^*) : \Lambda'_\infty \longrightarrow \Lambda_\infty$ generated by $\phi^* : C^\infty(N) \longrightarrow C^\infty(M)$ which, for simplicity, will be also denoted by ϕ^* as for ordinary forms. Then, locally

$$\phi^*(d_K y^\alpha) = d_K(\phi^*(y^\alpha)) = \sum_{\{J_1,\ldots,J_l\}} \frac{\partial^l \phi^\alpha}{\partial x^{\mu_1} \cdots \partial x^{\mu_l}} d_{J_1} x^{\mu_1} \wedge \cdots \wedge d_{J_l} x^{\mu_l} \in \Lambda_\infty,$$

where the sum runs over all repeated indexes and all partitions $\{J_1, \ldots, J_l\}$ of $K = \{k_1, \ldots, k_r\} \subset \mathbb{N}$ into l parts, $1 \leq l \leq r$.

4 Covariant tensors as iterated differential forms

For any $p \in \mathbb{N}$ the functor $\mathrm{D}_A^p \equiv \mathrm{D}_A \circ \cdots \circ \mathrm{D}_A : \mathbf{Mod}_A^g \longrightarrow \mathbf{Mod}_A^g$ is represented by the module $T_p^0(M) \equiv \Lambda^1(M)^{\otimes p}$ of covariant p–tensors on M. For any p the map

$$\Box : A \times \cdots \times A \ni (f_1, \ldots, f_p) \longmapsto d_1 f_1 \wedge \cdots \wedge d_p f_p \in \Lambda_\infty$$

is a multi–derivation, i.e. $\Box \in \mathrm{D}_A^p(\Lambda_\infty)$. Therefore, there exists a unique A–homomorphism $\iota_p : T_p^0(M) \longrightarrow \Lambda_\infty$ such that

$$\iota_p(df_1 \otimes \cdots \otimes df_p) = \Box(f_1, \ldots, f_p) = d_1 f_1 \wedge \cdots \wedge d_p f_p \in \Lambda_\infty$$

for any $f_1, \ldots, f_p \in A$.

Proposition 8 *ι_p is injective.*

Proof. Obvious from local expression. ■

Proposition 8 shows that the calculus of covariant tensors over M is just a part of differential calculus over the algebra Λ_∞ and, therefore, is not *conceptually closed*. In that sense the proposed embedding of tensors into Λ_∞ may be seen as a conceptual closure of tensor calculus.

Standard operations with tensors, such as multiplications, "permutations of indexes", insertions of vector fields, Lie derivatives, etc, have proper counterparts in Λ_∞ and this looks as follows.

First, all kinds of tensor multiplications are encoded in the wedge product "$\wedge$" in Λ_∞. For instance, exterior and symmetric products of differentials df and dg look as $d_1f \wedge d_2g - d_1g \wedge d_2f$ and $d_1f \wedge d_2g + d_1g \wedge d_2f$, respectively. Second, the embedding ι_p is equivariant with respect to the natural action τ_p of the permutation group S_p on $T_p^0(M)$, i.e., $\iota_p \circ \tau_p(\sigma) = \kappa_\sigma \circ \iota_p$, $\forall \sigma \in S_p$.

Evaluation of a tensor field $T \in T_p^0(M)$ on a p–ple of vector fields $X_1, \ldots, X_p \in \mathrm{D}(M)$ is interpreted in Λ_∞ by means of the formula

$$T(X_1, \ldots, X_p) = (i_{X_p}^{(p)} \circ \cdots \circ i_{X_1}^{(1)})(\iota_p(T)) \in A \subset \Lambda_\infty. \tag{3}$$

Indeed, if $T = df_1 \otimes \cdots \otimes df_p$, $f_1, \ldots, f_p \in C^\infty(M)$, then

$$\begin{aligned}(i_{X_p}^{(p)} \circ \cdots \circ i_{X_1}^{(1)})(\iota_p(T)) &= (i_{X_p}^{(p)} \circ \cdots \circ i_{X_1}^{(1)})(d_1f_1 \wedge \cdots \wedge d_pf_p) = X_1(f_1) \cdot \cdots \cdot X_p(f_p) \\ &= T(X_1, \ldots, X_p).\end{aligned}$$

In particular, the insertion of a vector field $X \in \mathrm{D}(M)$ into the l–th place of T is given by

$$\iota_p(T(\,\cdot\,, \ldots, \,\cdot\,, \underset{l}{X}, \,\cdot\,, \ldots, \,\cdot\,)) = (i_X^{(l)} \circ \iota_p)(T).$$

Similarly, the Lie derivative $\mathcal{L}_X T$ of a tensor field $T \in T_p^0(M)$ along a vector field $X \in \mathrm{D}(M)$ is given by

$$\iota_p(\mathcal{L}_X T) = (X \circ \iota_p)(T).$$

Indeed, let $T = \sum g_{\alpha_1 \cdots \alpha_p} df_{\alpha_1} \otimes \cdots \otimes df_{\alpha_p}$, $f_1, \ldots, f_p \in C^\infty(M)$. Then

$$\begin{aligned}(X \circ \iota_p)(T) &= X(\textstyle\sum g_{\alpha_1 \cdots \alpha_p} d_1f_{\alpha_1} \wedge \cdots \wedge d_pf_{\alpha_p}) = \sum X(g_{\alpha_1 \cdots \alpha_p}) d_1f_{\alpha_1} \wedge \cdots \wedge d_pf_{\alpha_p} \\ &+ \textstyle\sum g_{\alpha_1 \cdots \alpha_p} d_1X(f_{\alpha_1}) \wedge \cdots \wedge d_pf_{\alpha_p} + \cdots + \sum g_{\alpha_1 \cdots \alpha_p} d_1f_{\alpha_1} \wedge \cdots \wedge d_pX(f_{\alpha_p}) \\ &= \iota_p(\textstyle\sum X(g_{\alpha_1 \cdots \alpha_p}) df_{\alpha_1} \otimes \cdots \otimes df_{\alpha_p} + \sum g_{\alpha_1 \cdots \alpha_p} dX(f_{\alpha_1}) \otimes \cdots \otimes df_{\alpha_p} \\ &+ \cdots + \textstyle\sum g_{\alpha_1 \cdots \alpha_p} df_{\alpha_1} \otimes \cdots \otimes dX(f_{\alpha_p})) \\ &= \iota_p(\mathcal{L}_X T).\end{aligned}$$

The intrinsic characterization of covariant tensors as elements of Λ_∞ is as follows.

Proposition 9 *A homogeneous element* $\Omega \in \Lambda_p \subset \Lambda_\infty$ *of multi–degree* $(1,1,\ldots,1) \in \mathbb{Z}^p$ *is a covariant tensor on* M, *i.e.,* $\Omega \in \operatorname{im} \iota_p$, *iff the map*

$$\tilde{\Omega} : \mathrm{D}(M) \times \cdots \times \mathrm{D}(M) \ni (X_1, \ldots, X_p) \longmapsto (i^{(p)}_{X_p} \circ \cdots \circ i^{(1)}_{X_1})(\Omega) \in A \subset \Lambda_\infty$$

is A*–multi–linear.*

Proof. Let $\Omega \in \Lambda_p$ be a homogeneous element of multi–degree $(1,1,\ldots,1)$. Ω can be expressed in the form

$$\Omega = \iota_p(T) + \sum_{\{J_1,\ldots,J_l\}} g^{J_1\cdots J_l}_{\alpha_1\cdots\alpha_p} d_{J_1}(f_{\alpha_1 J_1}) \wedge \cdots \wedge d_{J_l}(f_{\alpha_l J_l})$$

where $T \in T^0_p(M)$, $g^{J_1\cdots J_l}_{\alpha_1\cdots\alpha_p}, f_{\alpha_1 J_1}, \ldots, f_{\alpha_l J_l} \in C^\infty(M)$ and the sum runs over $\alpha_1, \ldots, \alpha_l$ and all partitions $\{J_1, \ldots, J_l\}$ of $\{1, \ldots, p\}$ into l parts, $1 \leq l < p$. Let $X \in \mathrm{D}(M)$ and $s \leq p$. Without loss of generality suppose that $J_1 = \{i_1, \ldots, i_r, s\}$. Then,

$$\begin{aligned} i^{(s)}_X \Omega =& \iota_p(T(\,\cdot\,, \ldots, \,\cdot\,, \underset{s}{X}, \,\cdot\,, \ldots, \,\cdot\,)) \\ &+ \sum_{\{J_1,\ldots,J_l\}} g^{J_1\cdots J_l}_{\alpha_1\cdots\alpha_p} d_{\{i_1,\ldots,i_r\}}(X(f_{\alpha_1 J_1})) \wedge d_{J_2}(f_{\alpha_2 J_2}) \wedge \cdots \wedge d_{J_l}(f_{\alpha_l J_l}). \end{aligned}$$

Thus, $\tilde{\Omega}$ is multilinear iff $\sum_{\{J_1,\ldots,J_l\}} g^{J_1\cdots J_l}_{\alpha_1\cdots\alpha_p} d_{J_1}(f_{\alpha_1 J_1}) \wedge \cdots \wedge d_{J_l}(f_{\alpha_l J_l}) = 0$, i.e. $\Omega = \iota_p(T)$. ■

References

[1] A. M. Vinogradov, in M. Henneaux, I. Krasil'shchik, and A. Vinogradov (Eds.), *Secondary Calculus and Cohomological Physics*, Contemporary Mathematics **219**, AMS (1998). See also The Diffiety Inst. Preprint Series, DIPS 5/98.

[2] A. M. Vinogradov, *Cohomological Analysis of Partial Differential Equations and Secondary Calculus*, AMS "Translation of Mathematical Monographs "series **204** (2001).

[3] A. M. Vinogradov, *Soviet Math. Dokl.* **13** (1972) 1058.

[4] A. M. Vinogradov, *J. Soviet Math.* **17** (1981) 1624.

[5] M. M. Vinogradov, *Russian Math. Surveys* **44**, n° 3 (1989) 220.

[6] A. Verbovetsky, *J. Geom. Phys.* **18** (1996) 195.

[7] A. M. Vinogradov and L. Vitagliano, *Dokl. Math.* **73**, n° 2 (2006) 169.

[8] J. Nestruev, *Smooth Manifolds and Observables*, Springer–Verlag (New York) 2003.